Spy Game on Graphs

Nathann Cohen[1] Nícolas A. Martins[2] **Fionn Mc Inerney**[3]
Nicolas Nisse[3] Stéphane Pérennes[3] Rudini Sampaio[2]

[1]CNRS, Univ Paris Sud, LRI, Orsay, France

[2]Universidade Federal do Ceará, Fortaleza, Brazil

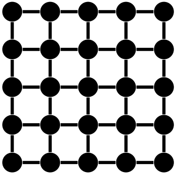
[3]Université Côte d'Azur, Inria, CNRS, I3S, France

Lyon, France, October 24, 2017

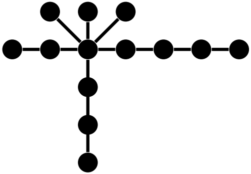
GAG Workshop

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



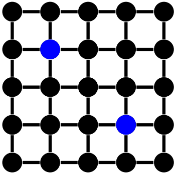
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



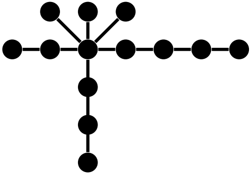
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



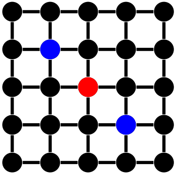
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



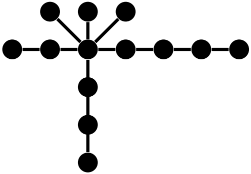
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,

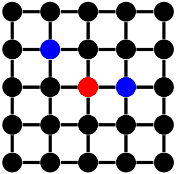


Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)

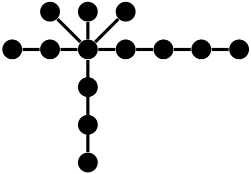


Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
    - Coordination for common goal, e.g.,



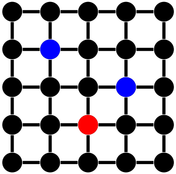
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



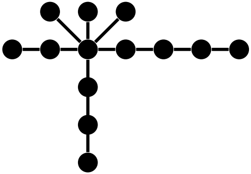
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
    - Coordination for common goal, e.g.,

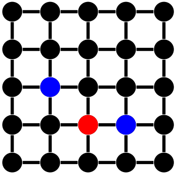


Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)

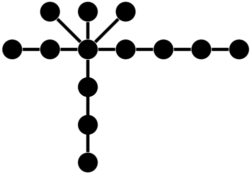


Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



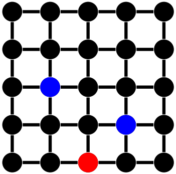
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



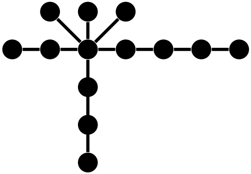
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



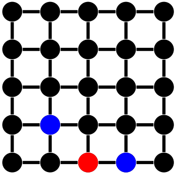
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



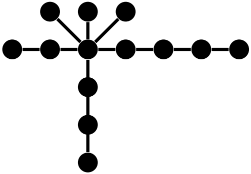
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



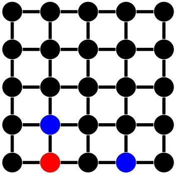
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



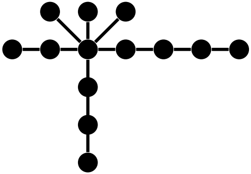
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



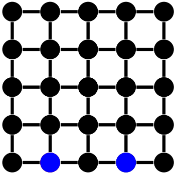
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



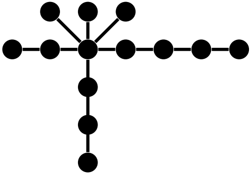
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



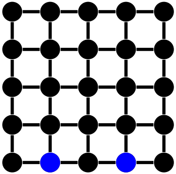
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



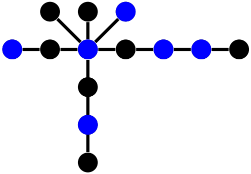
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



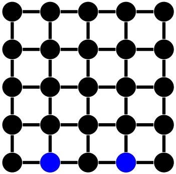
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



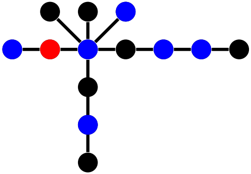
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



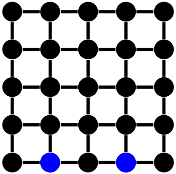
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



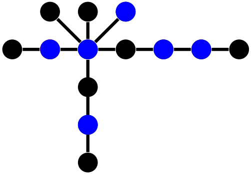
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
    - Coordination for common goal, e.g.,



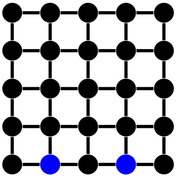
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



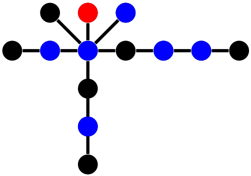
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



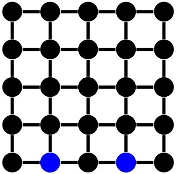
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



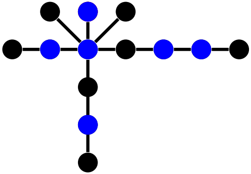
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



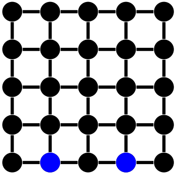
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



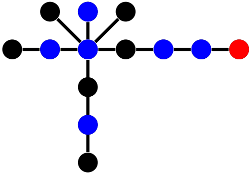
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



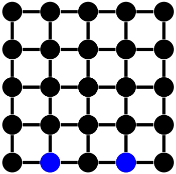
Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



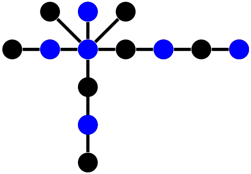
Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Pursuit-Evasion Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)



Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009)

Spy Game

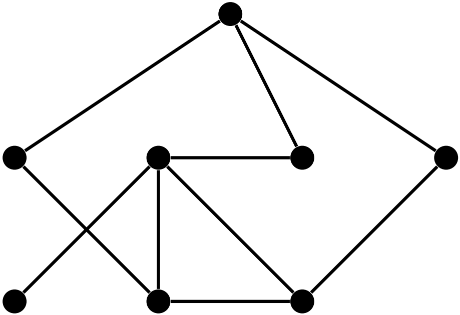
Spy (1^{st}) vs guards (2^{nd}) in a graph G.

Start : Spy placed at a vertex. Then, guards placed.

Turn-by-turn : Spy traverses up to $\mathbf{s} \geq 2$ edges. Guards traverse up to 1 edge.

Goal : Spy wants to be at least distance $\mathbf{d} + 1$ from all guards.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



Spy Game

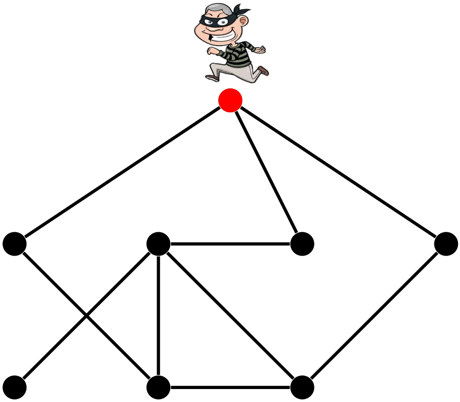
Spy (1^{st}) vs guards (2^{nd}) in a graph G.

Start : Spy placed at a vertex. Then, guards placed.

Turn-by-turn : Spy traverses up to $\mathbf{s} \geq 2$ edges. Guards traverse up to 1 edge.

Goal : Spy wants to be at least distance $\mathbf{d} + 1$ from all guards.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



Spy Game

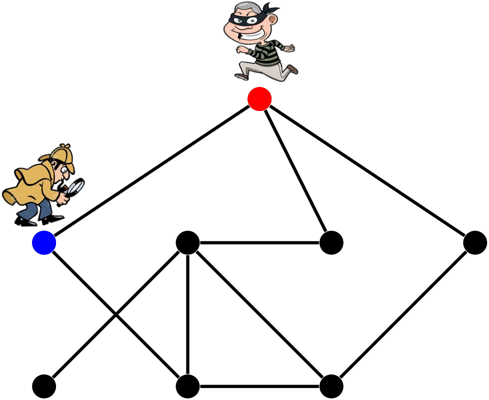
Spy (1^{st}) vs guards (2^{nd}) in a graph G.

Start : Spy placed at a vertex. Then, guards placed.

Turn-by-turn : Spy traverses up to $\mathbf{s} \geq 2$ edges. Guards traverse up to 1 edge.

Goal : Spy wants to be at least distance $\mathbf{d} + 1$ from all guards.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



Spy Game

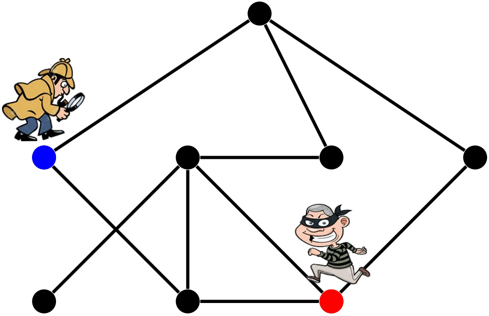
Spy (1^{st}) vs guards (2^{nd}) in a graph G.

Start : Spy placed at a vertex. Then, guards placed.

Turn-by-turn : Spy traverses up to $\mathbf{s} \geq 2$ edges. Guards traverse up to 1 edge.

Goal : Spy wants to be at least distance $\mathbf{d} + 1$ from all guards.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



Spy Game

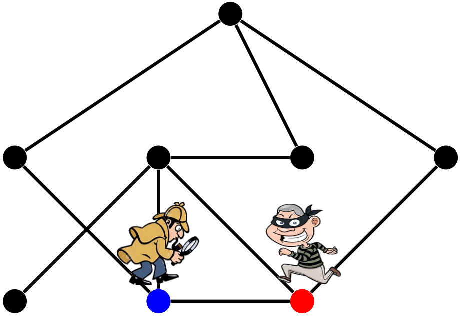
Spy (1^{st}) vs guards (2^{nd}) in a graph G.

Start : Spy placed at a vertex. Then, guards placed.

Turn-by-turn : Spy traverses up to $\mathbf{s} \geq 2$ edges. Guards traverse up to 1 edge.

Goal : Spy wants to be at least distance $\mathbf{d}+1$ from all guards.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



Spy Game

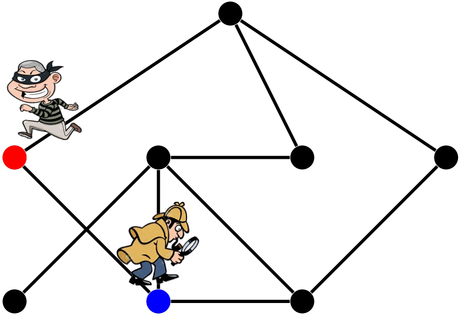
Spy (1^{st}) vs guards (2^{nd}) in a graph G.

Start : Spy placed at a vertex. Then, guards placed.

Turn-by-turn : Spy traverses up to $\mathbf{s} \geq 2$ edges. Guards traverse up to 1 edge.

Goal : Spy wants to be at least distance $\mathbf{d}+1$ from all guards.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



Spy Game

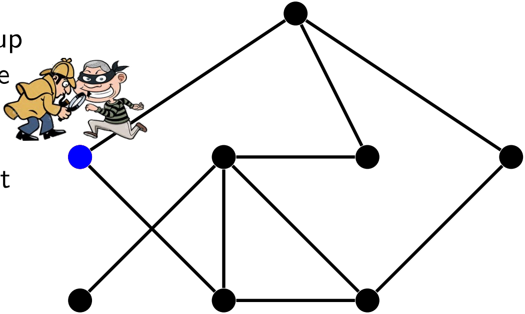
Spy (1^{st}) vs guards (2^{nd}) in a graph G.

Start : Spy placed at a vertex. Then, guards placed.

Turn-by-turn : Spy traverses up to $\mathbf{s} \geq 2$ edges. Guards traverse up to 1 edge.

Goal : Spy wants to be at least distance $\mathbf{d} + 1$ from all guards.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



Spy Game

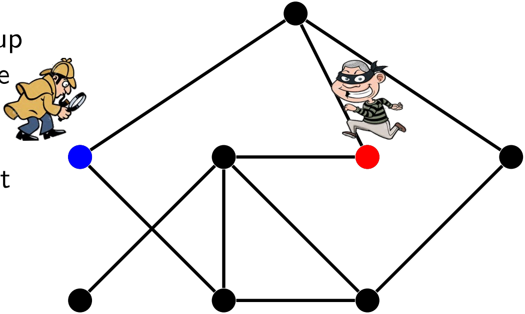
Spy (1^{st}) vs guards (2^{nd}) in a graph G.

Start : Spy placed at a vertex. Then, guards placed.

Turn-by-turn : Spy traverses up to $\mathbf{s} \geq 2$ edges. Guards traverse up to 1 edge.

Goal : Spy wants to be at least distance $\mathbf{d}+1$ from all guards.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



Spy Game

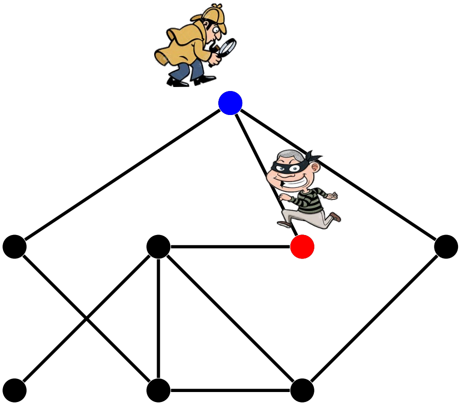
Spy (1^{st}) vs guards (2^{nd}) in a graph G.

Start : Spy placed at a vertex. Then, guards placed.

Turn-by-turn : Spy traverses up to $\mathbf{s} \geq 2$ edges. Guards traverse up to 1 edge.

Goal : Spy wants to be at least distance $\mathbf{d} + 1$ from all guards.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



Spy Game

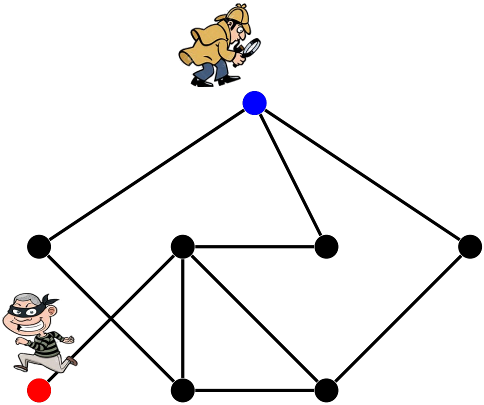
Spy (1^{st}) vs guards (2^{nd}) in a graph G.

Start : Spy placed at a vertex. Then, guards placed.

Turn-by-turn : Spy traverses up to $\mathbf{s} \geq 2$ edges. Guards traverse up to 1 edge.

Goal : Spy wants to be at least distance $\mathbf{d}+1$ from all guards.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



Guard Number : $gn_{s,d}(G)$

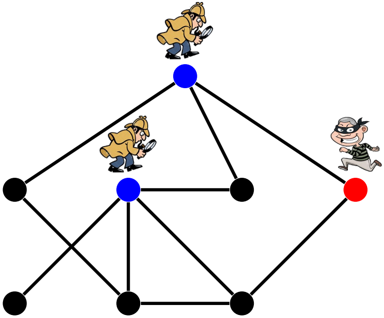
Definition

For all $s \geq 2$, $d \geq 0$ and a graph G, $gn_{s,d}(G)$ is the minimum number of guards guaranteed to win vs the spy.

Guard Number : $gn_{s,d}(G)$

Definition

For all $s \geq 2$, $d \geq 0$ and a graph G, $gn_{s,d}(G)$ is the minimum number of guards guaranteed to win vs the spy.



$$gn_{2,1}(G) = 2$$

$$gn_{s,1}(G) \leq \gamma(G)$$

Our Results : Computing gn

Complexity

Calculating $gn_{s,d}$ is NP-hard in general.

Tight bounds for paths

$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+q} \right\rceil$ where $q = \lfloor \frac{2d}{s-1} \rfloor$.

Almost tight bounds for cycles

$\left\lceil \frac{n+2q}{2(d+q)+3} \right\rceil \leq gn_{s,d}(C_n) \leq \left\lceil \frac{n+2q}{2(d+q)+1} \right\rceil$ where $q = \lfloor \frac{2d}{s-1} \rfloor$.

Polynomial time Linear Program for trees

Can calculate $gn_{s,d}(T)$ and a corresp. strategy in polynomial time.

Grids

$\exists \beta > 0$, s.t. $\Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n \times n})$.

Related Work

- Cops vs robber (capture at a distance) (Bonato et al, 2010).

Related Work

- Cops vs robber (capture at a distance) (Bonato et al, 2010).
- Cops vs fast robber (Fomin et al, 2010).

Related Work

- Cops vs robber (capture at a distance) (Bonato et al, 2010).
- Cops vs fast robber (Fomin et al, 2010).
  - How many cops needed in an $n \times n$ grid ?

Related Work

- Cops vs robber (capture at a distance) (Bonato et al, 2010).
- Cops vs fast robber (Fomin et al, 2010).
    - How many cops needed in an $n \times n$ grid ?
- Eternal Domination (Goddard et al, 2005).

Related Work

- Cops vs robber (capture at a distance) (Bonato et al, 2010).
- Cops vs fast robber (Fomin et al, 2010).
  - How many cops needed in an $n \times n$ grid ?
- Eternal Domination (Goddard et al, 2005).
  - $\gamma^m(m \times n \ grid) \leq \lceil \frac{mn}{5} \rceil + O(m+n)$ (Lamprou et al, 2016).

Related Work

- Cops vs robber (capture at a distance) (Bonato et al, 2010).
- Cops vs fast robber (Fomin et al, 2010).
  - How many cops needed in an $n \times n$ grid ?
- Eternal Domination (Goddard et al, 2005).
  - $\gamma^m(m \times n\ grid) \leq \lceil \frac{mn}{5} \rceil + O(m+n)$ (Lamprou et al, 2016).
  - $\gamma^m(G) = gn_{s,d}(G)$ when $s = \infty$ and $d = 0$.

Paths

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,
$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$

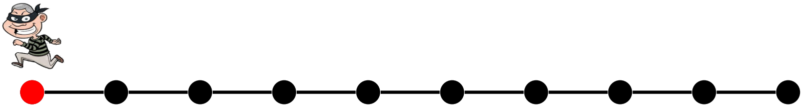
Paths : Lower bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.



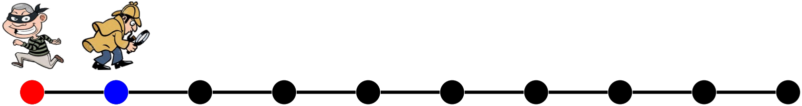
Paths : Lower bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.



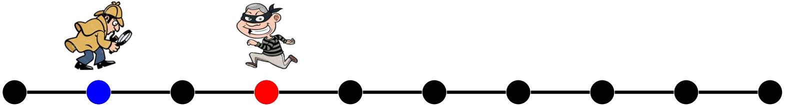
Paths : Lower bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.



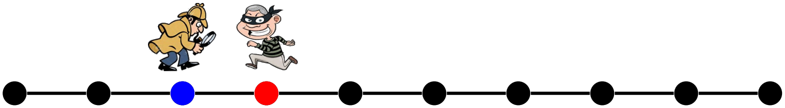
Paths : Lower bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.



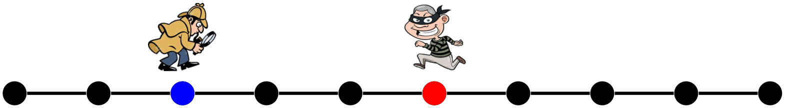
Paths : Lower bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.

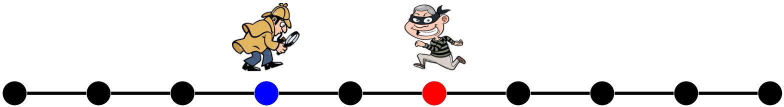


Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.



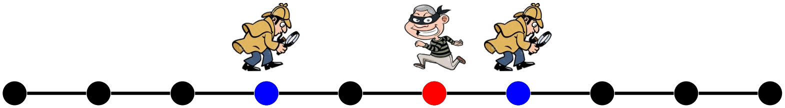
Paths : Lower bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.



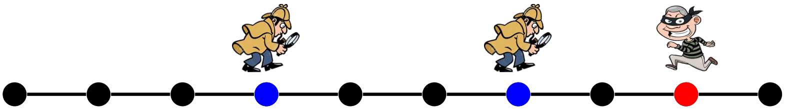
Paths : Lower bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.



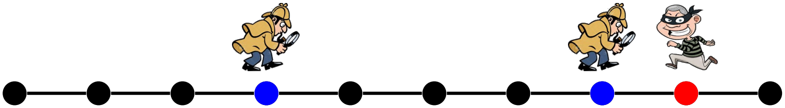
Paths : Lower bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.

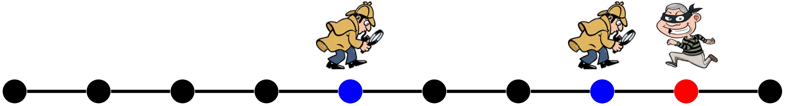


Paths : Lower bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,
$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$

Ex : $s = 3$ and $d = 1$.



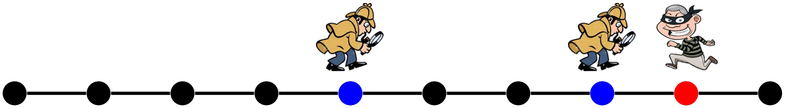
Paths : Lower bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.



$$gn_{3,1}(P_{10}) = 2$$

Paths : Upper bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,
$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$

Ex : $s = 3$ and $d = 1$.

1 guard can protect subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.

Paths : Upper bound

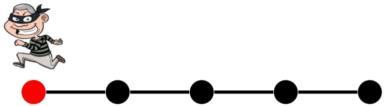
Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.

1 guard can protect subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



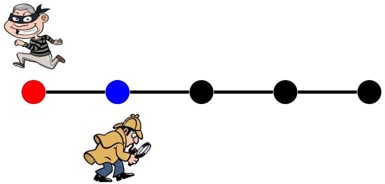
Paths : Upper bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,
$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$

Ex : $s = 3$ and $d = 1$.

1 guard can protect subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.

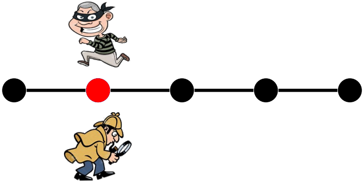


Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,
$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$

Ex : $s = 3$ and $d = 1$.

1 guard can protect subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



Paths : Upper bound

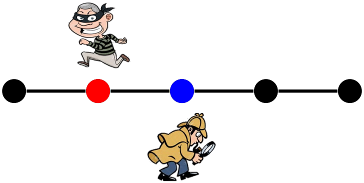
Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$

Ex : $s = 3$ and $d = 1$.

1 guard can protect subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



Paths : Upper bound

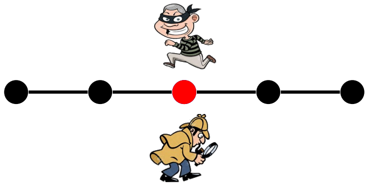
Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.

1 guard can protect subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



Paths : Upper bound

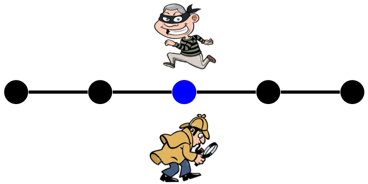
Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.

1 guard can protect subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



Paths : Upper bound

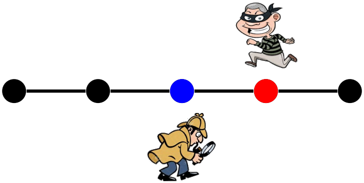
Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.

1 guard can protect subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



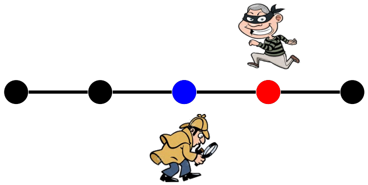
Paths : Upper bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,
$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$

Ex : $s = 3$ and $d = 1$.

1 guard can protect subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



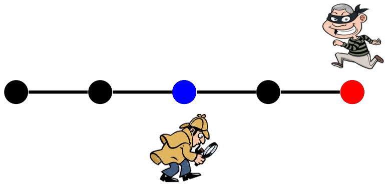
Paths : Upper bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,
$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.

1 guard can protect subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



Paths : Upper bound

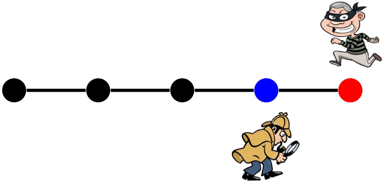
Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex : $s = 3$ and $d = 1$.

1 guard can protect subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



Paths : Upper bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,
$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$

Each guard protects a subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.

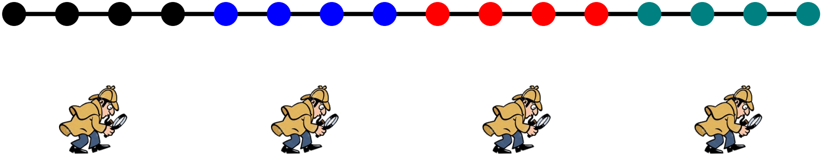
Paths : Upper bound

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor\frac{2d}{s-1}\rfloor} \right\rceil$$

Each guard protects a subpath of $2d + 2 + \lfloor\frac{2d}{s-1}\rfloor$ vertices.



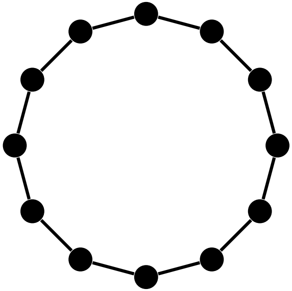
Cycles : Upper Bound Case $2d < s - 1$

Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $2d < s - 1$, and a cycle C_n on n vertices, $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

Ex : $s = 6$ and $d = 0$.

$$gn_{6,0}(C_{12}) = 4$$

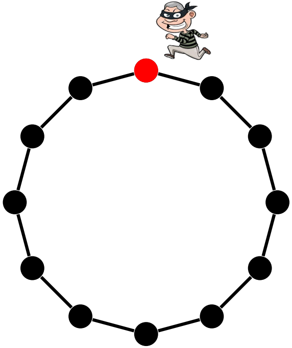


Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $2d < s - 1$, and a cycle C_n on n vertices, $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

Ex : $s = 6$ and $d = 0$.

$$gn_{6,0}(C_{12}) = 4$$



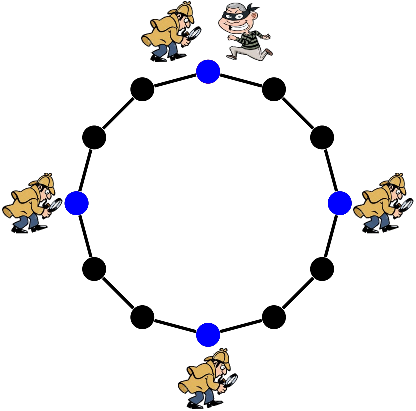
Cycles : Upper Bound Case $2d < s - 1$

Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $2d < s - 1$, and a cycle C_n on n vertices, $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

Ex : $s = 6$ and $d = 0$.

$$gn_{6,0}(C_{12}) = 4$$



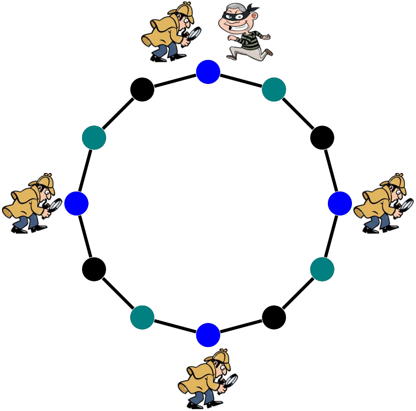
Cycles : Upper Bound Case $2d < s - 1$

Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $2d < s - 1$, and a cycle C_n on n vertices, $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

Ex : $s = 6$ and $d = 0$.

$$gn_{6,0}(C_{12}) = 4$$



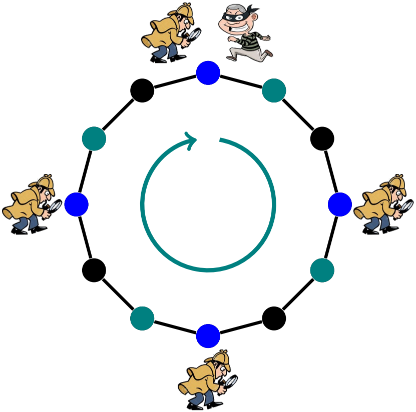
Cycles : Upper Bound Case $2d < s - 1$

Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $2d < s - 1$, and a cycle C_n on n vertices, $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

Ex : $s = 6$ and $d = 0$.

$$gn_{6,0}(C_{12}) = 4$$

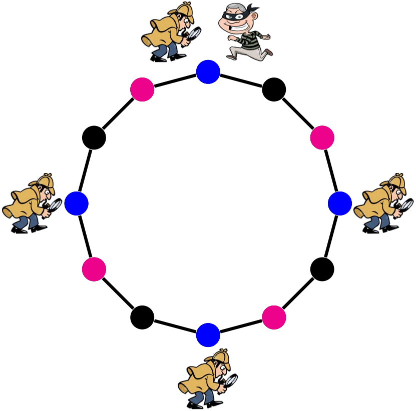


Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $2d < s - 1$, and a cycle C_n on n vertices, $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

Ex : $s = 6$ and $d = 0$.

$$gn_{6,0}(C_{12}) = 4$$



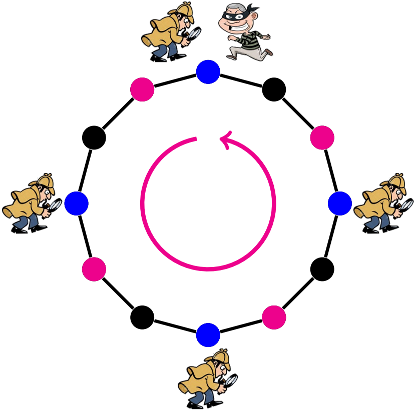
Cycles : Upper Bound Case $2d < s - 1$

Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $2d < s - 1$, and a cycle C_n on n vertices, $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

Ex : $s = 6$ and $d = 0$.

$$gn_{6,0}(C_{12}) = 4$$



Cycles

Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $q = 0$, and a cycle C_n on n vertices, $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

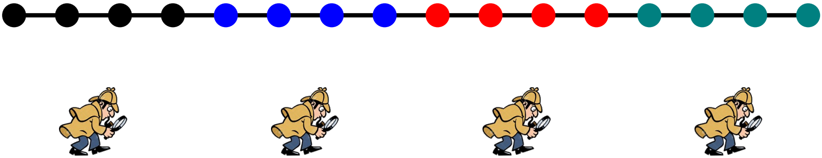
Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $q \neq 0$, and a cycle C_n on n vertices, $\left\lceil \frac{n+2q}{2(d+q)+3} \right\rceil \leq gn_{s,d}(C_n) \leq \left\lceil \frac{n+2q}{2(d+q)+1} \right\rceil$.

Reminder : $q = \left\lfloor \frac{2d}{s-1} \right\rfloor$.

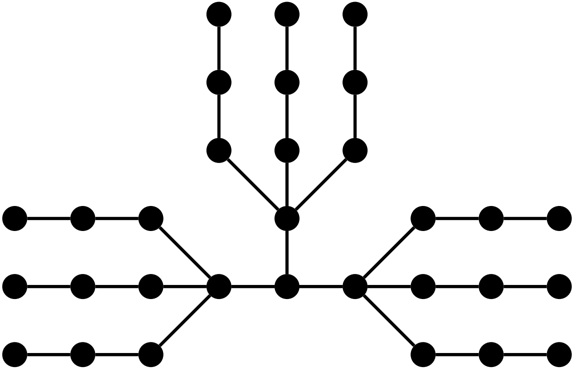
Trees are Harder

Paths : 1 guard per subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



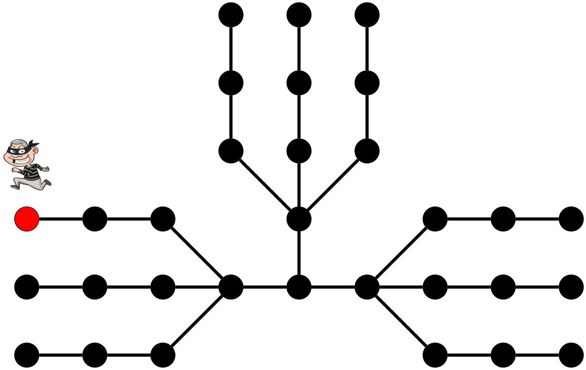
Trees are Harder

Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

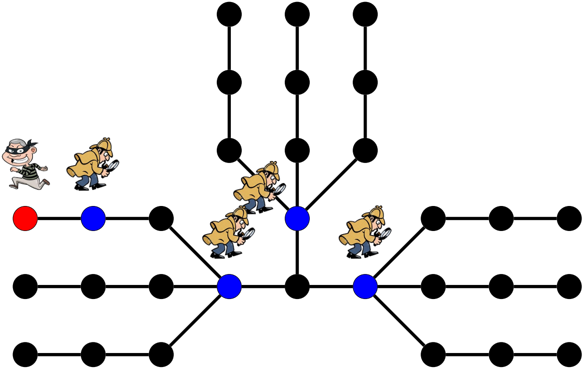
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

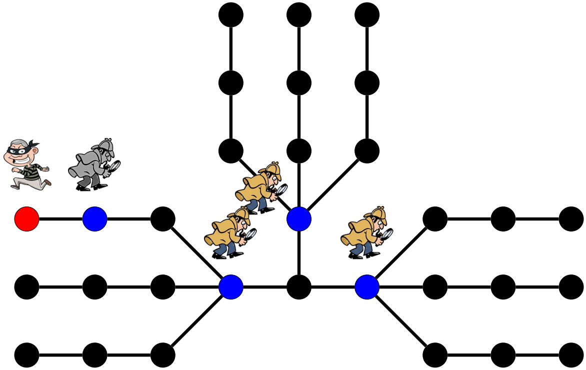
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

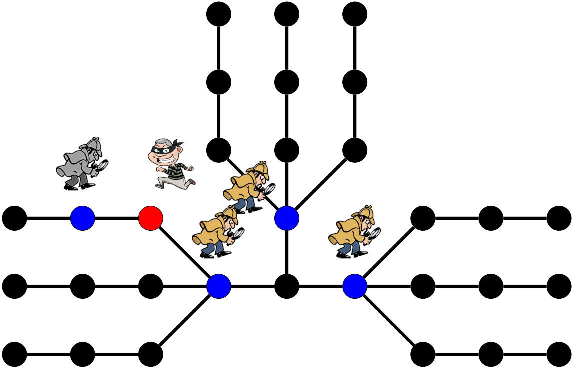
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

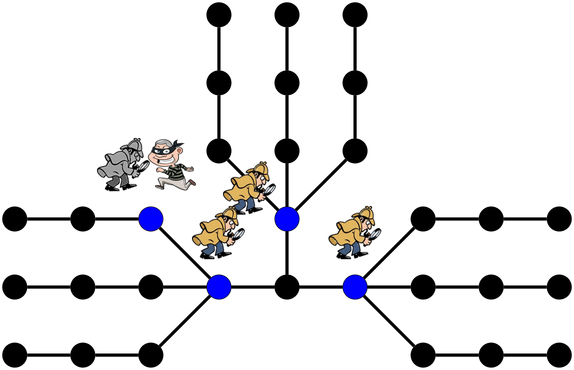
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

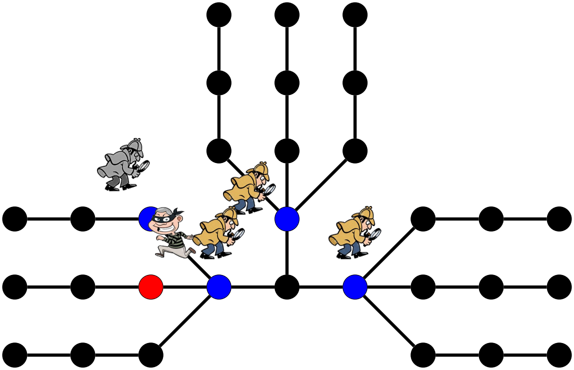
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

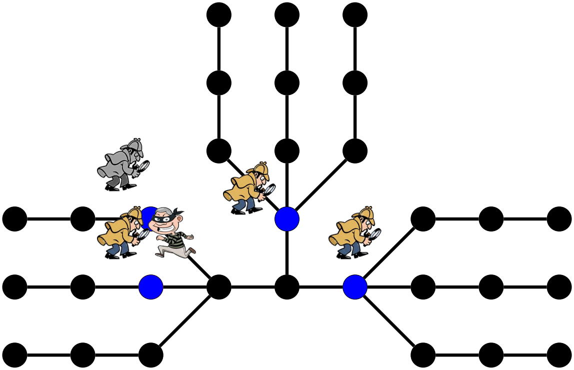
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

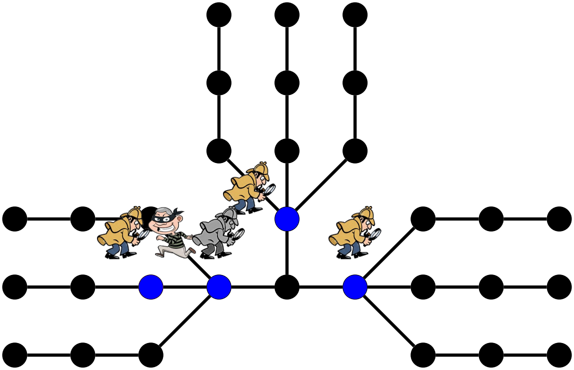
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

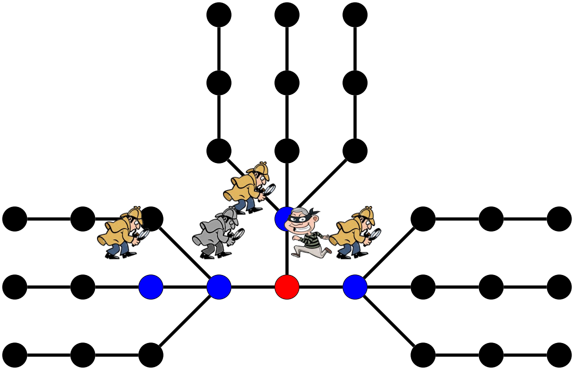
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

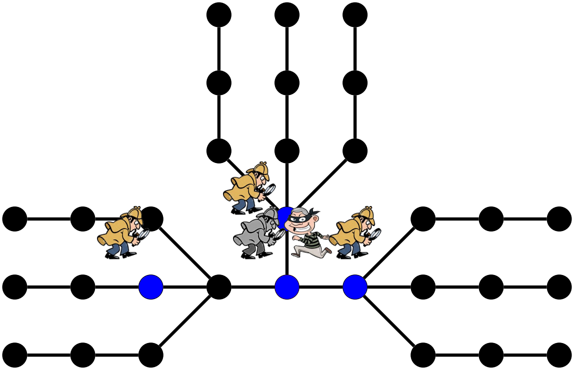
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

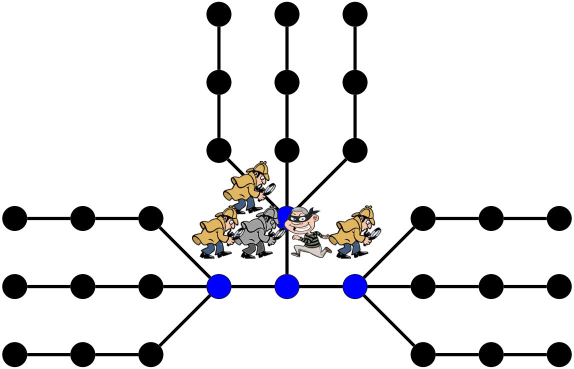
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

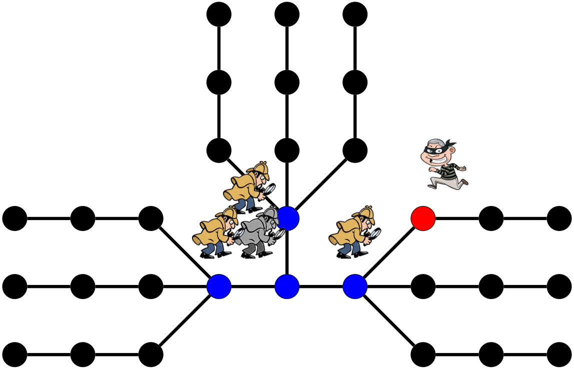
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

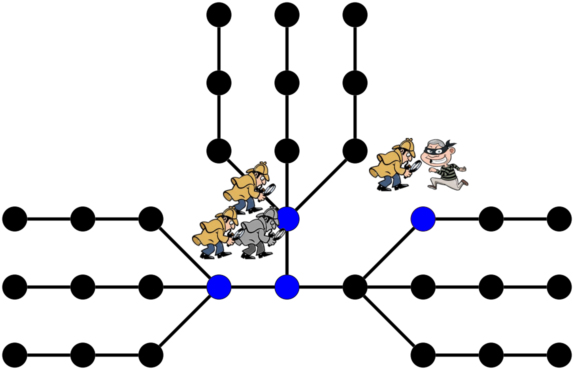
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

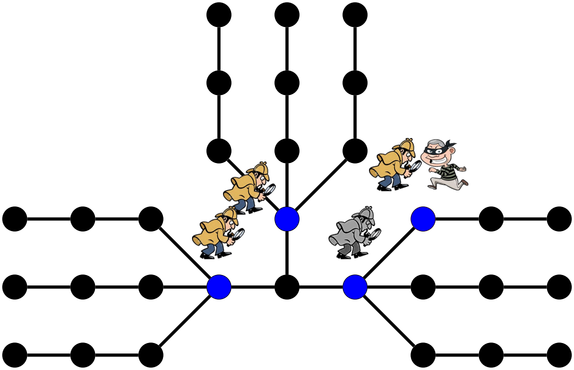
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

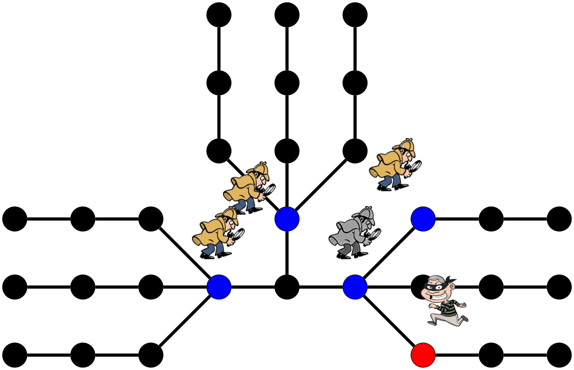
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

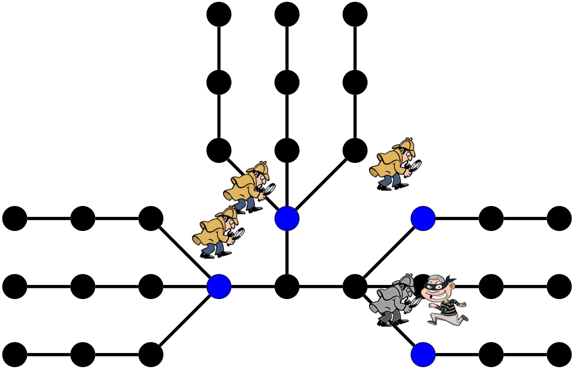
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

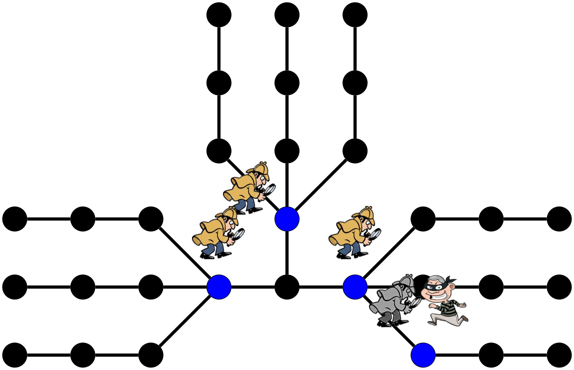
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

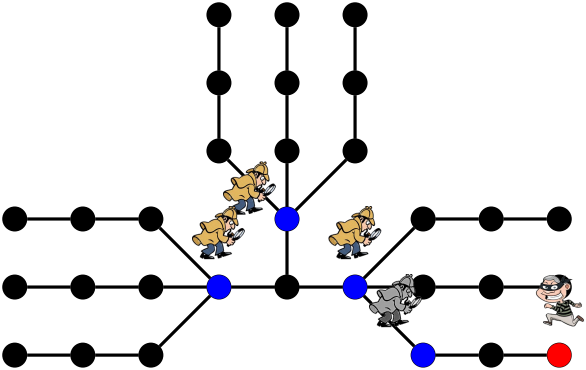
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

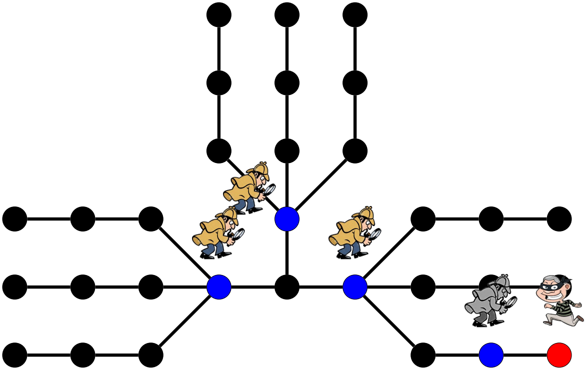
Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

Trees are Harder

Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

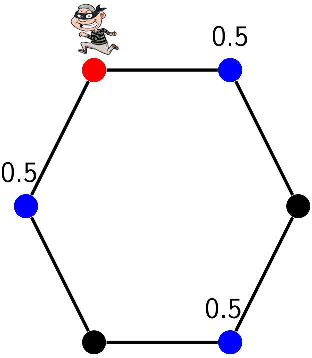
Fractional Version of the Game

- Guards may be fractional entities ; movements rep. by flows.
- Unchanged for spy. Total fraction of guards distance $\leq d$ from spy must be ≥ 1.

Fractional Version of the Game

- Guards may be fractional entities ; movements rep. by flows.
- Unchanged for spy. Total fraction of guards distance $\leq d$ from spy must be ≥ 1.

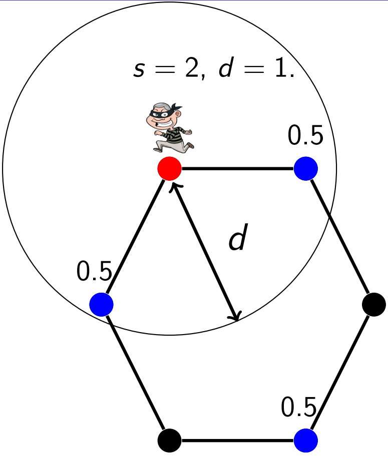
$s = 2,\ d = 1.$



$gn_{2,1}(C_6) = 2$
but
1.5 guards suffice.

Fractional Version of the Game

- Guards may be fractional entities ; movements rep. by flows.
- Unchanged for spy. Total fraction of guards distance $\leq d$ from spy must be ≥ 1.

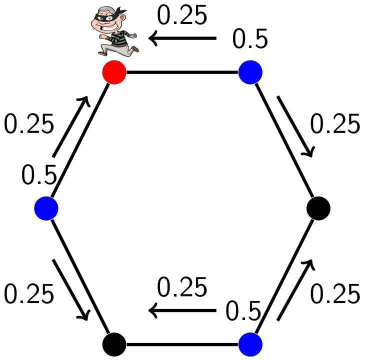


$gn_{2,1}(C_6) = 2$
but
1.5 guards suffice.

Fractional Version of the Game

- Guards may be fractional entities ; movements rep. by flows.
- Unchanged for spy. Total fraction of guards distance $\leq d$ from spy must be ≥ 1.

$s = 2,\ d = 1.$

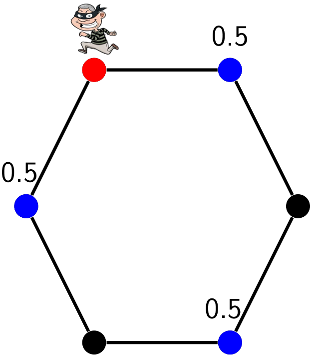


$gn_{2,1}(C_6) = 2$
but
1.5 guards suffice.

Fractional Version of the Game

- Guards may be fractional entities ; movements rep. by flows.
- Unchanged for spy. Total fraction of guards distance $\leq d$ from spy must be ≥ 1.
- Linear program to compute optimal fractional strategy.

$s = 2,\ d = 1.$



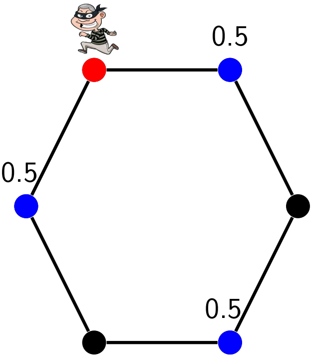
$gn_{2,1}(C_6) = 2$
but
1.5 guards suffice.

Fractional Version of the Game

- Guards may be fractional entities ; movements rep. by flows.
- Unchanged for spy. Total fraction of guards distance $\leq d$ from spy must be ≥ 1.

- Linear program to compute optimal fractional strategy.
- Optimal fractional strategy $\Rightarrow$ optimal integral strategy in trees.

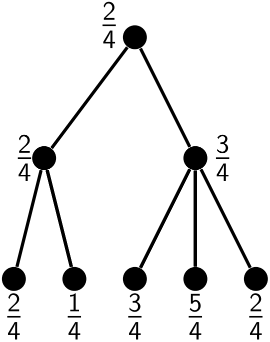
$s = 2$, $d = 1$.



$gn_{2,1}(C_6) = 2$
but
1.5 guards suffice.

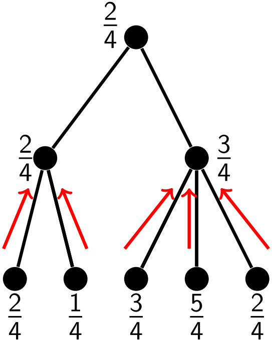
Opt. Fractional Strategy ⇒ Opt. Integral Strategy in Trees

Theorem : Can transform optimal fractional strategy into optimal integral strategy in polynomial time.



Fractional Conf.

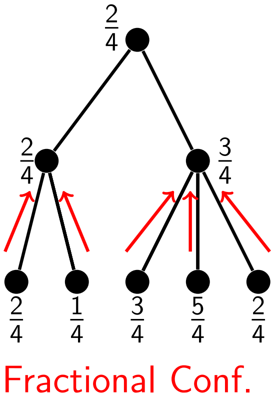
Theorem : Can transform optimal fractional strategy into optimal integral strategy in polynomial time.

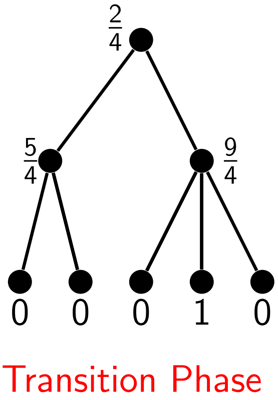


Fractional Conf.

Opt. Fractional Strategy ⇒ Opt. Integral Strategy in Trees

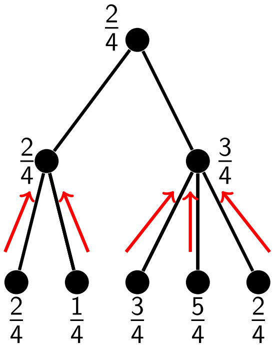
Theorem : Can transform optimal fractional strategy into optimal integral strategy in polynomial time.

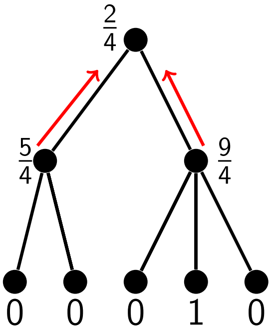




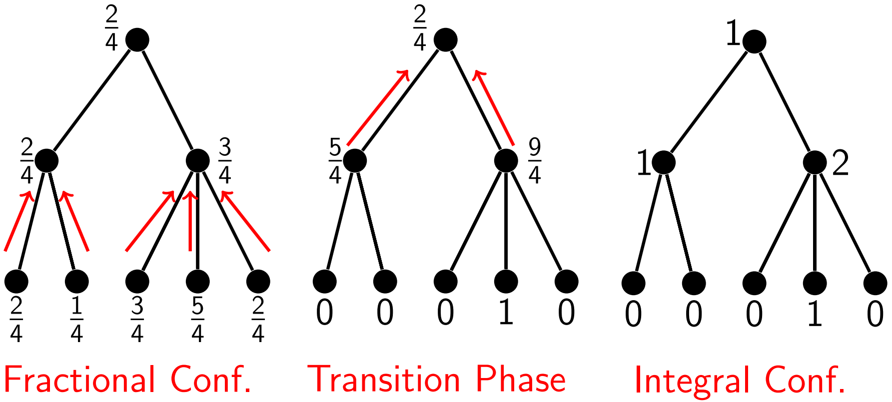
Opt. Fractional Strategy ⇒ Opt. Integral Strategy in Trees

Theorem : Can transform optimal fractional strategy into optimal integral strategy in polynomial time.



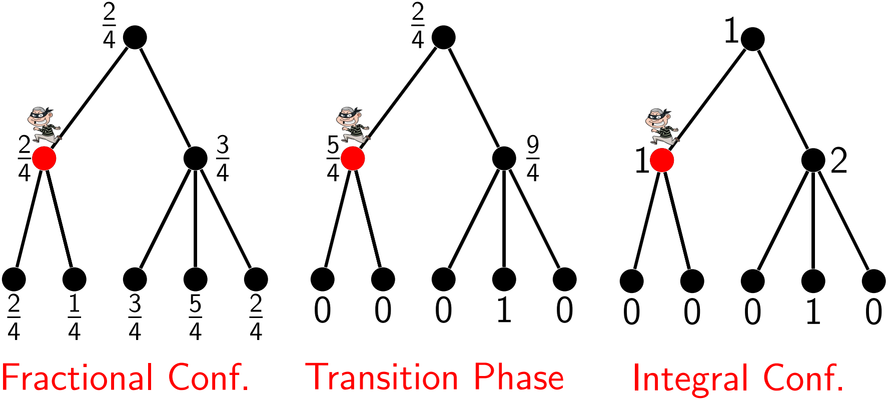


Theorem : Can transform optimal fractional strategy into optimal integral strategy in polynomial time.



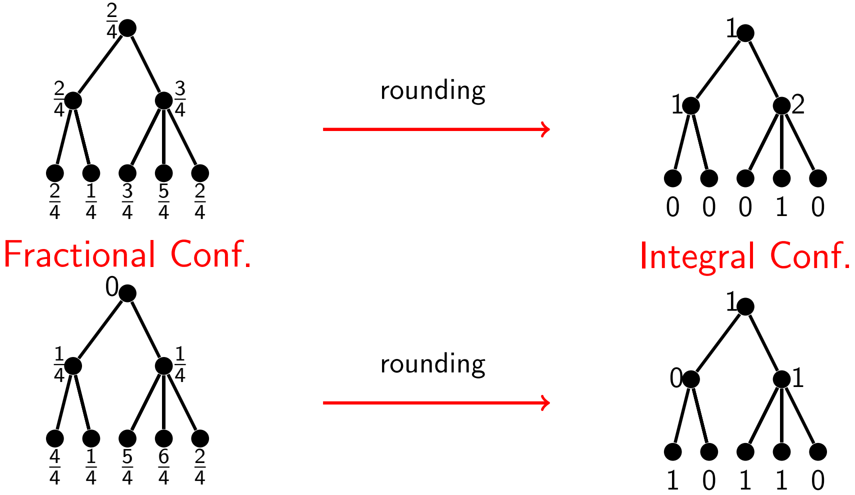
Opt. Fractional Strategy ⇒ Opt. Integral Strategy in Trees

Theorem : Can transform optimal fractional strategy into optimal integral strategy in polynomial time.



Tree's protection and guards' movements preserved.

Opt. Fractional Strategy $\Rightarrow$ Opt. Integral Strategy in Trees



Tree's protection and guards' movements preserved.

Restricted Strategies

$f : V^k \times V \Rightarrow V^k$ (Unrestricted strategy)

$\omega : V \Rightarrow V^k$ (Restricted strategy)

Restricted Strategies

$f : V^k \times V \Rightarrow V^k$ (Unrestricted strategy)

$\omega : V \Rightarrow V^k$ (Restricted strategy)

- Guards' positions depend only on position of spy.
- 1 Unique configuration for guards for each position of spy.

Restricted Strategies

$f : V^k \times V \Rightarrow V^k$ (Unrestricted strategy)

$\omega : V \Rightarrow V^k$ (Restricted strategy)

- Guards' positions depend only on position of spy.
- 1 Unique configuration for guards for each position of spy.

Theorem

Optimal fractional strategy $\Rightarrow$ optimal fractional restricted strategy in trees.

Restricted Strategies

$f : V^k \times V \Rightarrow V^k$ (Unrestricted strategy)

$\omega : V \Rightarrow V^k$ (Restricted strategy)

- Guards' positions depend only on position of spy.
- 1 Unique configuration for guards for each position of spy.

Theorem

Optimal fractional strategy $\Rightarrow$ optimal fractional restricted strategy in trees.

Can calculate optimal restricted fractional strategies with Linear Program in polynomial time.

Linear Program to Compute Restricted Strategy

Restricted strategy : $\omega : V \Rightarrow V^k$

$\omega_{x,u}$: quantity of guards on u when spy is on x.

$f_{x,x',u,u'}$: quantity of guards that go from u to u' when spy goes from x to x'.

Linear Program to Compute Restricted Strategy

Restricted strategy : $\omega : V \Rightarrow V^k$

$\omega_{x,u}$: quantity of guards on u when spy is on x.

$f_{x,x',u,u'}$: quantity of guards that go from u to u' when spy goes from x to x'.

$$(1) \text{ Minimize } \sum_{v \in V} \omega_{x_0,v}$$

Minimize number of guards.

Linear Program

Restricted strategy : $\omega : V \Rightarrow V^k$

$\omega_{x,u}$: quantity of guards on u when spy is on x.

$f_{x,x',u,u'}$: quantity of guards that go from u to u' when spy goes from x to x'.

$$(2) \sum_{v \in N_d[x]} \omega_{x,v} \geq 1 \qquad \forall x \in V$$

Guarantees always at least 1 guard within distance d of spy.

Linear Program

Restricted strategy : $\omega : V \Rightarrow V^k$

$\omega_{x,u}$: quantity of guards on u when spy is on x.

$f_{x,x',u,u'}$: quantity of guards that go from u to u' when spy goes from x to x'.

$$(3) \sum_{u' \in N[u]} f_{x,x',u,u'} = \omega_{x,u} \qquad \forall u \in V,\ x' \in N_s[x]$$

$$(4) \sum_{u' \in N[u]} f_{x,x',u',u} = \omega_{x',u} \qquad \forall u \in V,\ x' \in N_s[x]$$

Guarantees validity of moves of guards when spy moves.

Linear Program

Restricted strategy : $\omega : V \Rightarrow V^k$

$\omega_{x,u}$: quantity of guards on u when spy is on x.

$f_{x,x',u,u'}$: quantity of guards that go from u to u' when spy goes from x to x'.

$$(3) \sum_{u' \in N[u]} f_{x,x',u,u'} = \omega_{x,u} \qquad \forall u \in V,\, x' \in N_s[x]$$

$$(4) \sum_{u' \in N[u]} f_{x,x',u',u} = \omega_{x',u} \qquad \forall u \in V,\, x' \in N_s[x]$$

Guarantees validity of moves of guards when spy moves.

$O(n^4)$ real variables and constraints.

Main Result : *gn* in Trees

Theorem

$\forall s > 1$, $d \geq 0$ and all trees T, $gn_{s,d}(T)$ and a corresponding strategy can be calculated in polynomial time.

Idea of proof : Linear Program can compute opt. frac. restr. strategy in polynomial time.

Run LP. From previous theorem, strategy is opt. frac.

Can transform opt. frac. into opt. int. in polynomial time.

Grids

Theorem

$\exists\beta > 0$, s.t. $\forall s > 1$, $d \geq 0$, $\Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n\times n})$.

Idea of proof : Lower bound holds for fractional version.

Grids

Theorem

$\exists \beta > 0$, s.t. $\forall s > 1$, $d \geq 0$, $\Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n\times n})$.

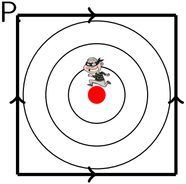
Idea of proof : Lower bound holds for fractional version.

Torus and grid have same order of number of guards.

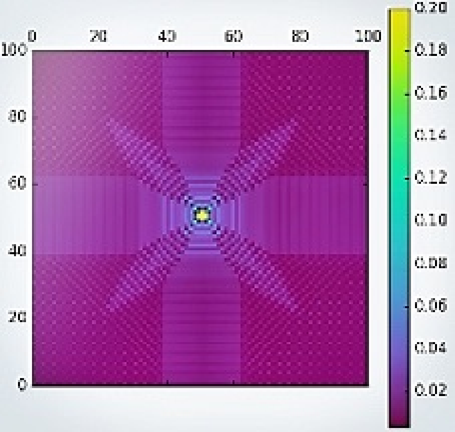
Theorem

$\exists \alpha \geq log(3/2) \approx 0.58$, s.t. $\forall s > 1$, $d \geq 0$,
$fgn_{s,d}(G_{n\times n}) \leq O(n^{2-\alpha})$.

Idea of proof : Density function $\omega^*(v) = \frac{c}{(dist(v,v_0)+1)^{log 3/2}}$ for a constant $c > 0$ satisfies LP.



Distribution of Guards in the Torus for an optimal symmetrical spy-positional strategy when $n = 100$, $m = 100$, $s = 2$ and $d = 1$



Further Work

- Determine $gn_{s,d}(G_{n\times n})$.
- Approximate $gn_{s,d}(G)$ in polynomial time in certain classes of graphs ?
- Fractional approach applied to other combinatorial games.

Thanks !